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Inertial effects in time-dependent motion of thin films and drops

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Capillarity is an important feature in controlling the spreading of liquid drops and in the coating of substrates by liquid films. For thin films and small contact angles, lubrication theory enables the analysis of the motion to be reduced to single evolution equations for the heights of the drops or films, provided the inertia of the liquid can be neglected. In general, the presence of inertia destroys the major simplification provided by lubrication theory, but two special cases that can be treated are identified here. In the first example, the approach of a drop to its equilibrium position is studied. For sufficiently low Reynolds numbers, the rate of approach to the terminal state and the contact angle are slightly reduced by inertia, but, above a critical Reynolds number, the approach becomes oscillatory. In the latter case there is no simple relation connecting the dynamic contact angle and contact-line speed. In the second example, the spreading drop is supported by a plate that is forced to oscillate in its own plane. For the parameter range considered, the mean spreading is unaffected by inertia, but the oscillatory motion of the contact line is reduced in magnitude as inertia increases, and the drop lags behind the plate motion. The oscillatory contact angle increases with inertia, but is not in phase with the plate oscillation.

1. Introduction

The theoretical treatment of the spreading of a liquid drop on a solid and the coating of a solid surface with a liquid film is greatly simplified if the drop and film are thin, so that the lubrication approximation can be employed. When this approximation is valid, it is possible to determine the velocity field of the liquid as a functional of height, and the problem reduces to the solution of a nonlinear evolution equation for the height profile of the drop or film. To leading order, at low speeds, the dynamics is controlled by a balance among capillarity, gravity and viscosity, and the liquid inertia does not enter. Considerable success has been achieved by this approach to the solution of this class of problems (see Oron, Davis & Bankoff 1997).

These solutions are limited to speeds low enough to give small capillary and Reynolds numbers. The extension of the theory to higher speeds introduces inertia into the problem, and, even in the case of thin drops and films, the analyses become much more difficult. The great simplification previously found by the application of lubrication theory no longer exists; instead, the system is governed by the coupling of a nonlinear partial differential equation for the velocity field and an evolution equation for the height profile.

It is possible, however, to find a class of problems in which inertial effects can be assessed without losing the simplification of lubrication theory, thus allowing one to

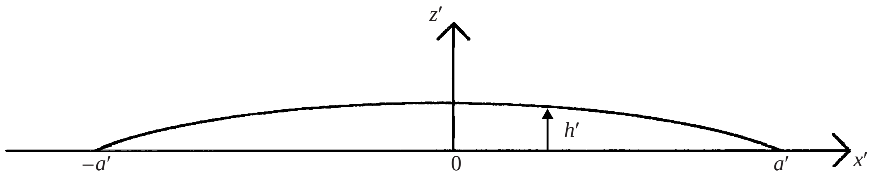

FIGURE 1. Sketch of the spreading drop.

learn of new behaviours. The acceleration of the liquid is composed of two parts: the local acceleration and the convective acceleration. In certain cases, the first part is much greater than the second, and, because the nonlinearity of the convective part does not then enter the equation to leading order, a linear equation for the velocity still holds. A familiar example of this linearization of inertia is provided by the Stokes layer in boundary-layer theory.

Some examples of time-dependent liquid motion governed by capillarity, viscosity and this particular influence of inertia are examined in this paper. For simplicity, gravity is ignored and the motions are taken to be two-dimensional. The formulation of the examples to be considered are described in § 2, which also contains a discussion of the damping of capillary waves on the free surface of a thin horizontal film of liquid, and the response of the layer to a forced lateral oscillation of the supporting plane. These elementary examples are included because they relate to the more complicated situation found in the succeeding sections. In § 3, the approach of a spreading drop to its terminal, static, shape is studied. It is found that this approach may be monotonic or oscillatory. In § 4, the spreading of a drop on a solid plate, which is undergoing rapid lateral oscillations, is discussed. Section 5 is devoted to a discussion of the effect of inertia on apparent contact angles.

In the examples examined in §§ 3 and 4, there is a moving contact line at the edge of the drop. It is therefore necessary to make certain assumptions concerning both the removal of the force-singularity at the edge and the contact angle at the edge. In this paper, it is assumed that there is a small amount of slip near the edge, and that the microscopic contact angle remains fixed (independent of speed) at its value determined by the balance of intermolecular forces. Any dynamic behaviour of the contact angle is then consequent upon the dynamics of the liquid close to the edge, where large stresses result in a large variation in the slope of the free surface.

For inertialess motions, the analysis of the slip region by Hocking & Rivers (1982) and Cox (1986) entails a three-layer matching between the inner slip region, the outer region, and the intermediate region where there is a local wedge solution. In a posthumous paper by Cox (1998), the first attempt to extend the analysis of spreading to large Reynolds numbers was made. His predictions, and their relation to the role of inertia identified in §§ 3 and 4 are discussed in § 5.

2. Formulation

2.1. *Spreading drop*

A liquid drop of width $2a'(t')$ on a stationary plane at $z' = 0$ has its free surface given by $z' = h'(x', t')$, where x' and z' are measured tangentially and normally, respectively, and the drop is symmetrical, so that $h(\pm a', t') = 0$ (see figure 1).

When $h' \ll a'$ one can simplify the governing equations using lubrication arguments and at the same time retain the inertia:

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + w'\frac{\partial u'}{\partial z'} = -\frac{\partial p'}{\rho \partial x'} + \nu \frac{\partial^2 u'}{\partial z'^2}, \tag{2.1a}$$

$$0 = \frac{\partial p'}{\rho \partial z'}, \tag{2.1b}$$

$$\frac{\partial u'}{\partial x'} + \frac{\partial w'}{\partial z} = 0, \tag{2.1c}$$

where u' and w' are the tangential and normal velocity components, p' is the pressure relative to the atmospheric pressure, ρ is the density, and ν is the kinematic viscosity of the liquid. The boundary conditions for these equations are as follows: no penetration of liquid into the plate and slip on the plate,

$$w' = 0, \qquad u' = \lambda' \frac{\partial u'}{\partial z'} \quad \text{at} \quad z' = 0; \tag{2.2a}$$

zero shear stress, the kinematic condition and Laplace's relation on the liquid–gas interface,

$$\frac{\partial u'}{\partial z'} = 0, \qquad \frac{\partial h'}{\partial t'} + u'\frac{\partial h'}{\partial x'} - w' = 0, \qquad p' = -\sigma \frac{\partial^2 h'}{\partial x'^2} \quad \text{at} \quad z' = h'; \tag{2.2b}$$

symmetry conditions at the centreline,

$$u' = 0, \qquad \frac{\partial h'}{\partial x'} = \frac{\partial^3 h'}{\partial x'^3} = 0 \quad \text{at} \quad x' = 0; \tag{2.2c}$$

zero height and a contact-angle condition at the contact line,

$$h' = 0, \qquad \frac{\partial h'}{\partial x'} = \mp \theta_s \quad \text{at} \quad x' = \pm a'. \tag{2.2d}$$

A linear slip model with slip coefficient λ' has been assumed. The surface tension, σ, and the advancing microscopic contact angle, θ_s, are assumed to be constant. Conservation of mass requires that, for all t',

$$\int_0^{a'} h' \, \mathrm{d}x' = A', \tag{2.3}$$

where A' is half the cross-sectional area of the drop. From (2.1b) and the pressure boundary condition in (2.2b), it follows that, independently of z',

$$p' = -\sigma \frac{\partial^2 h'}{\partial x'^2}. \tag{2.4}$$

It is helpful to introduce a stream function $\psi'(x', z', t')$ defined by

$$u' = \frac{\partial \psi'}{\partial z'}, \quad w' = -\frac{\partial \psi'}{\partial x'}. \tag{2.5}$$

To non-dimensionalize these variables, tangential and normal length scales a_0 and h_0 are chosen, and the non-dimensional (unprimed) variables are defined by

$$\left.\begin{array}{l} h' = h_0 h, \quad z' = h_0 z, \quad x' = a_0 x, \quad a' = a_0 a, \\ u' = U u, \quad w' = U(h_0/a_0) w, \quad \psi' = U h_0 \psi, \quad t' = (a_0/U) t; \\ p' = \rho U^2 p, \quad A' = a_0 h_0 A. \end{array}\right\} \tag{2.6}$$

The length a_0 can be chosen as the radius of the drop in equilibrium. For lubrication theory to hold, the contact angle θ_s must be small, and h_0 can defined so that it is equal to $\theta_s a_0$. The speed U balances viscous forces and surface tension,

$$U = \frac{\sigma}{\rho\nu}\left(\frac{h_0}{a_0}\right)^3. \tag{2.7}$$

The equilibrium solution to which the drop tends, as t increases, is then given by

$$h = \tfrac{1}{2}(1-x^2), \quad a = 1, \quad A = \tfrac{1}{3}. \tag{2.8}$$

The non-dimensional forms of equations (2.1) and (2.3) are given by

$$\frac{\partial^3\psi}{\partial z^3} = -\frac{\partial^3 h}{\partial x^3} + \frac{1}{\varepsilon}\left(\frac{\partial^2\psi}{\partial z\partial t} + \frac{\partial\psi}{\partial z}\frac{\partial^2\psi}{\partial x\partial z} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial z^2}\right), \tag{2.9a}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}[\psi(x,h,t)] = 0, \qquad \int_0^a h\,\mathrm{d}x = \tfrac{1}{3}, \tag{2.9b}$$

and the boundary conditions (2.2*a–d*) become

$$\psi = 0, \qquad \frac{\partial\psi}{\partial z} = \lambda\frac{\partial^2\psi}{\partial z^2} \quad \text{at} \quad z = 0; \qquad \frac{\partial^2\psi}{\partial z^2} = 0 \quad \text{at} \quad z = h, \tag{2.10a}$$

$$\psi = \frac{\partial^2\psi}{\partial x^2} = \frac{\partial h}{\partial x} = \frac{\partial^3 h}{\partial x^3} = 0 \quad \text{at} \quad x = 0, \tag{2.10b}$$

$$h = 0, \qquad \frac{\partial h}{\partial x} = -1 \quad \text{at} \quad x = a. \tag{2.10c}$$

The parameters in these equations are the scaled slip length λ and the reciprocal Reynolds number ε, defined by

$$\lambda = \frac{\lambda'}{h_0}, \qquad \varepsilon = \frac{1}{Re} = \frac{\nu}{Uh_0}\left(\frac{a_0}{h_0}\right). \tag{2.11}$$

Note that in the viscous-dominated case, $\varepsilon \to \infty$ and the contact-line speed is proportional to $1/\ln(1/\lambda)$, which tends to zero as $\lambda \to 0$. λ is numerically small and the asymptotic limit, $\lambda \to 0$, gives a convenient simplification even for finite ε.

2.2. *Thin film on a stationary plate*

The inertial effects on the spreading of a drop will influence both its central portion and the edge regions. Before considering the complete problem, it is instructive to examine the effect of inertia on small disturbances to a thin liquid film, which will relate to the central region of the drop. The equilibrium state is given by $h = 1$, and for small-amplitude waves, the height and stream function can be written in the form

$$h = 1 + \alpha\exp(\mathrm{i}\delta x)\exp(-st), \qquad \psi = \alpha\psi_1(z)\exp(\mathrm{i}\delta x)\exp(-st), \tag{2.12}$$

where α is the small amplitude of the wave. In this problem, there are no contact lines and hence slip is not necessary and the edge conditions can be dropped. The lubrication approximation is valid if the dimensional wavelength is large compared to the depth of the film, i.e. $2\pi/\delta \gg h_0/a_0$. The relevant parts of equations (2.9) and boundary conditions (2.10), with the parameters (2.11), are then given by

$$\left.\begin{aligned} &\frac{\mathrm{d}^3\psi_1}{\mathrm{d}z^3} = \mathrm{i}\delta^3 - \frac{s}{\varepsilon}\frac{\mathrm{d}\psi_1}{\mathrm{d}z}, \qquad -s + \mathrm{i}\delta\psi_1(1) = 0, \\ &\psi_1 = 0, \qquad \frac{\mathrm{d}\psi_1}{\mathrm{d}z} = 0 \quad \text{at} \quad z = 0, \qquad \frac{\mathrm{d}^2\psi_1}{\mathrm{d}z^2} = 0 \quad \text{at} \quad z = 1. \end{aligned}\right\} \tag{2.13}$$

For waves that do not grow in amplitude s has a non-negative real part. The solution of these equations is straightforward, and provides the equation that determines s as a function of wavelength and Reynolds number,

$$-\frac{s}{\delta^4} = \frac{q - \tan q}{q^3}, \quad q^2 = \frac{s}{\varepsilon}. \tag{2.14}$$

When $\varepsilon \to 0$, $s \sim \pm \mathrm{i}\sqrt{(\varepsilon\delta^4)}$, and when $\varepsilon \to \infty$, $s \sim \delta^4/3$. Without viscosity, the waves are not damped, and when viscosity is large, the temporal oscillations disappear and the wave amplitudes decrease monotonically to zero.

The roots of (2.14) can be found in terms of a parameter r, defined by

$$r = \frac{\varepsilon}{\delta^4} = \frac{\tan q - q}{q^5}. \tag{2.15}$$

For r greater than a critical value r_c, there are two real positive values of s, which, for large r, are given approximately by

$$r = \frac{1}{3q^2}, \qquad s = \tfrac{1}{3}\delta^4, \quad \text{and} \quad q \to \tfrac{1}{2}\pi, \qquad s \to \tfrac{1}{4}\pi^2\varepsilon. \tag{2.16}$$

At the critical value r_c these two roots coalesce, and the critical quantities are given approximately by

$$r_c = 0.5367, \quad q_c = 1.1127, \quad s_c = 1.2381\varepsilon = 0.6645\delta^4. \tag{2.17}$$

For values of r less than r_c, q and s are complex numbers, and, as $r \to 0$ ($Re \to \infty$),

$$q \sim \exp(\mp\tfrac{1}{4}\mathrm{i}\pi)/r^{1/4}, \quad s \sim \pm \mathrm{i}\varepsilon/r^{1/2}. \tag{2.18}$$

A similar change from monotonic to oscillatory damping when the Reynolds number passes a critical value is found also in the solution for the spreading drop in § 3. This indicates that the oscillatory behaviour of the contact line in that problem is associated with the presence of capillary waves on the free surface of the drop.

2.3. *Thin film on an oscillating plate*

If the plate undergoes a forced oscillation of frequency ω, with its velocity proportional to $\exp(\mathrm{i}\omega t)$, no capillary waves are generated. The oscillation is transmitted to the film through a Stokes layer, and the complex velocity u at the free surface is given by

$$u = \operatorname{sech}[\ell(1 + \mathrm{i})] \exp(\mathrm{i}\omega t), \quad \ell = \sqrt{\omega/2\varepsilon}. \tag{2.19}$$

The amplitude of the free-surface speed is given by

$$|u| = \left(\frac{2}{\cosh 2\ell + \cos 2\ell}\right)^{1/2}. \tag{2.20}$$

When $\ell = 0$, there is no inertia and the liquid moves with the plate. As ℓ increases, the speed decreases and tends to zero, with $|u| \sim 2\exp(-\ell)$. These results suggest that, for a drop placed on an oscillating plate, the spreading of the drop will be unaffected by the oscillation in the inertia-free limit, but that, when the effect of inertia is included, the oscillatory component of the speed of the edge of the drop will be less than that of the plate. The spreading of a drop on an oscillating plate is examined in § 4.

3. The approach to the terminal state for a spreading drop

In examining the spreading of a liquid drop, the objectives are to find the speed of the contact line, $\mathrm{d}a/\mathrm{d}t$ as a function of the radius of the drop, and of the parameters ε and λ, and to determine the dynamic behaviour of the contact angle.

At $\varepsilon = \infty$, the solution in the central region can be written symbolically as

$$h(x,t) = \frac{1}{2a^3}(a^2 - x^2) + a_t h_1, \quad \psi = a_t \psi_1(x,z,t), \tag{3.1}$$

provided $a_t = \mathrm{d}a/\mathrm{d}t$ is small in magnitude. Here a_t indicates an order of magnitude, not a function of time. For finite ε, this solution gives the leading term since the inertia terms in (2.9a) can be neglected, that is, since $|a_t|/\varepsilon \ll 1$. The solution, as given by Hocking (1982) and translated into the present notation, has the form

$$\frac{1}{a^6} - 1 = \ln\left(\frac{2}{3\lambda \mathrm{e}^2}\right)\frac{\mathrm{d}a}{\mathrm{d}t}, \quad \frac{\theta_{app}}{\theta_s} = \frac{1}{a^2}, \tag{3.2}$$

where θ_{app} is the leading term in the slope of the drop at the boundary of the outer region.

Thus, without inertia, a_t is proportional to $1/\ln(1/\lambda)$, and inertia only enters when $\varepsilon \ln(1/\lambda)$ is of unit order. From (2.9) and (3.1), the equations for ψ_1 and h_1 are

$$\frac{\partial^3 \psi_1}{\partial z^3} = -\frac{\partial^3 h_1}{\partial x^3} + \frac{a_t}{\varepsilon}\left(\frac{\partial^2 \psi_1}{\partial z \partial t} + \frac{\partial \psi_1}{\partial z}\frac{\partial^2 \psi_1}{\partial x \partial z} - \frac{\partial \psi_1}{\partial x}\frac{\partial^2 \psi_1}{\partial z^2}\right), \tag{3.3a}$$

$$-\frac{a^2 - 3x^2}{2a^4} + \frac{\partial}{\partial x}\psi_1\left(x, \frac{a^2 - x^2}{2a^3}, t\right) = 0. \tag{3.3b}$$

Equation (3.3b) can be integrated to give

$$-\frac{x(a^2 - x^2)}{2a^4} + \psi_1\left[x, \frac{1}{2a^3}(a^2 - x^2), t\right] = 0, \tag{3.4}$$

where the boundary condition $\psi_1 = 0$ at $x = 0$ has been used. Thus, as ε is reduced, the inertial terms first become significant when ε is of order a_t, and the equation for ψ_1 is no longer linear, and would require numerical solution. It would, of course, be possible to expand the solution in powers of a small parameter related to a_t/ε, but this would represent only a small change to the viscous solution and not be of much significance

The radius of the drop approaches its terminal value exponentially. When the radius is close to its terminal value, ψ_1 is exponentially small and the quadratic inertia terms in (3.3) will be much smaller than the linear ones. Hence, it is possible to examine this limit theoretically, and to determine some influence of inertia on the spreading of the drop.

For the approach to the equilibrium state, the radius and height of the drop and the stream function can be written as

$$a = 1 - \alpha \exp(-st), \quad h = \frac{(a^2 - x^2)}{2a^3} + \alpha s \exp(-st) h_1(x), \quad \psi = \alpha s \exp(-st)\psi_1(x,z), \tag{3.5}$$

where s has a positive real part and α measures the amplitude of the perturbation. The leading terms in the governing equations (2.9), are given by

$$\frac{\partial^3 \psi_1}{\partial z^3} = -\frac{\mathrm{d}^3 h_1}{\mathrm{d}x^3} - \frac{s}{\varepsilon}\frac{\partial \psi_1}{\partial z}\{1 + O[\alpha \exp(-st)]\}, \tag{3.6a}$$

$$\psi_1[x, \tfrac{1}{2}(1-x^2)] + \tfrac{1}{2}x(1-x^2) + O[(\alpha\exp(-st))] = 0, \tag{3.6b}$$

where s is small since ε is small and equation (2.9*b*) has been integrated once. The boundary conditions (2.10) and the mass conservation requirement become

$$\left.\begin{array}{l} \psi_1 = 0, \quad \dfrac{\partial\psi_1}{\partial z} = \lambda\dfrac{\partial^2\psi_1}{\partial z^2} \quad \text{at} \quad z = 0, \quad \dfrac{\partial^2\psi_1}{\partial z^2} = 0 \quad \text{at} \quad z = h_e \\ \psi_1 = \dfrac{\partial^2\psi_1}{\partial z^2} = \dfrac{\mathrm{d}h_1}{\mathrm{d}x} = \dfrac{\mathrm{d}^3h_1}{\mathrm{d}x^3} = 0 \quad \text{at} \quad x = 0, \quad h_1 = 0 \quad \text{at} \quad x = 1, \quad \displaystyle\int_0^1 h_1\,\mathrm{d}x = 0, \end{array}\right\} \tag{3.7}$$

where $h_e = \frac{1}{2}(1-x^2)$. Finally, the slope condition at the edge becomes

$$-\frac{1}{a^2} + as\exp(-st)\frac{\partial h_1}{\partial x} = -1 \quad \text{at} \quad x = a = 1 - \alpha\exp(-st). \tag{3.8}$$

Hence, to leading order, the condition becomes

$$\frac{\partial h_1}{\partial x} = -\frac{2}{s} \quad \text{at} \quad x = 1. \tag{3.9}$$

These equations and conditions must now be solved to determine s in terms of ε and λ.

The decay parameter s may be real or complex. If s is real and positive, a real positive quantity q can be defined by $q^2 = s/\varepsilon$. Then the solution of (3.6*a*) has the form

$$\psi_1 = -\frac{z}{q^2}\frac{\mathrm{d}^3h_1}{\mathrm{d}x^3} + A\sin qz + B(1-\cos qz), \tag{3.10}$$

where A and B are constants, determined by the boundary conditions. Equation (3.6*b*) then provides the differential equation for h_1 in the form

$$\frac{\mathrm{d}^3h_1}{\mathrm{d}x^3} = -\frac{xq^3h_e(1-\lambda q\tan qh_e)}{\tan qh_e - qh_e(1-\lambda qh_e)}, \tag{3.11}$$

with equations (3.7) and (3.8),

$$\frac{\mathrm{d}h_1}{\mathrm{d}x} = 0 \quad \text{at} \quad x = 0; \qquad \int_0^1 h_1\,\mathrm{d}x = 0; \qquad h_1 = 0, \qquad \frac{\mathrm{d}h_1}{\mathrm{d}x} = -\frac{2}{s} \quad \text{at} \quad x = 1. \tag{3.12}$$

In the central region, away from a vicinity of the contact line, slip can be ignored, but it must be included near the contact line to remove the singularity that would otherwise occur. Near the contact line, the variables can be scaled by writing

$$x = 1 - \lambda X, \quad h_1 = \lambda H(X), \tag{3.13}$$

and the leading term in (3.11) and the appropriate boundary conditions (3.7) and (3.8) become

$$\frac{\mathrm{d}^3H}{\mathrm{d}X^3} = \frac{3}{X(X+3)}; \quad H = 0, \quad \frac{\mathrm{d}H}{\mathrm{d}X} = \frac{2}{s} \quad \text{at} \quad X = 0; \quad \frac{\mathrm{d}^2H}{\mathrm{d}X^2} \to 0 \quad \text{as} \quad X \to \infty. \tag{3.14}$$

The last condition is needed to ensure that a match with the outer solution is possible. The solution is straightforward, and shows that

$$\frac{\mathrm{d}H}{\mathrm{d}X} \sim \frac{2}{s} - 3 + 3\ln 3 - 3\ln X, \quad X \gg 1. \tag{3.15}$$

In the central region, the equation for h_1, neglecting terms of order λ, is

$$\frac{\mathrm{d}^3 h_1}{\mathrm{d}x^3} = -\frac{x q^3 h_e}{\tan q h_e - q h_e}; \quad \frac{\mathrm{d}h_1}{\mathrm{d}x} = 0 \quad \text{at} \quad x = 0; \quad \int_0^1 h_1 \, \mathrm{d}x = 0; \quad h_1 = 0 \quad \text{at} \quad x = 1, \tag{3.16}$$

with $h_e = \frac{1}{2}(1 - x^2)$. In order to match with the inner solution,

$$\frac{\mathrm{d}h_1}{\mathrm{d}x} \sim -\frac{2}{s} + 3 - 3\ln 3 - 3\ln\lambda + 3\ln(1-x) \quad \text{as} \quad x \to 1. \tag{3.17}$$

From (3.16), it follows that

$$\frac{\mathrm{d}^3 h_1}{\mathrm{d}x^3} \sim \frac{-3}{(1-x)^2} \quad \text{as} \quad x \to 1, \tag{3.18}$$

so that it is possible to match the logarithmic terms in the outer and inner solutions. In the spreading problem without inertia, and when the drop is not close to its terminal size, such matching is not possible, and an intermediate region must be introduced (Hocking 1982).

The three conditions in (3.16) suffice to determine the three arbitrary constants that appear due to three integrations, so that the value of h_1 is determined. The slope condition (3.17) then fixes the value of s as a function of q. The integrand in (3.16) has poles at $x = 1$ and at those values of x for which $\tan q h_e = q h_e$. It is convenient first to subtract the singular part, which can be integrated exactly, and then to integrate numerically the regular integrand remaining. The result of this integration and the matching with (3.17) leads to the determination of a function $F(q)$, with

$$-\frac{2}{s} - 3\ln\lambda = F(q), \quad \varepsilon = \frac{s}{q^2}. \tag{3.19}$$

For a given λ and for a chosen value of q, this equation determines the pair of quantities, s and $\varepsilon = 1/Re$.

When $q = 0$ ($\varepsilon = \infty$), the integration can be performed exactly, and $F(0) = 6 + 3\ln(3/2) = 7.2164$. As q is increased, the numerical values of F increase and ε decreases. At a critical value of q, dependent on the slip coefficient λ, a minimum value of ε is reached, so that there is no solution of the form chosen for smaller values of ε (larger values of Re). For $\lambda = \exp(-15) = 3.06 \times 10^{-7}$, the critical values are $q = 16.34, 1/\varepsilon = Re = 2283$. The calculated values of s for this value of λ are shown as curve 1 in figure 2. The relevant part of the graph extends from the left to the point marked X1. For $\lambda = \exp(-10)$ the critical value of Re is 809, and for $\lambda = \exp(-20)$ it is 4728.

The failure of the solution at the critical Reynolds number suggests a bifurcation in which the decay parameter takes complex values. For complex $q = q_r + \mathrm{i}q_i$, there are no poles of the integrand on the x-axis, so it is not necessary to subtract these singularities. The solution procedure is to fix q_r, and to search for the value of q_i for which the calculated complex value of s is such that $s/q^2 = \varepsilon$ is real. The calculated values of s_r are shown as curve 2 in figure 2, extending from the point marked X2 to the right. There are no complex solutions for $Re < 1226$. Between the two points X1 and X2, curve 2 lies slightly above curve 1. Thus the complex-s solution bifurcates from the real-s solution at X2, and there are two possible solutions for $1226 < Re < 2283$. The value of s_i as a function of Re is also shown in figure 2. The bifurcation to complex values of s may be related to the presence of capillary waves

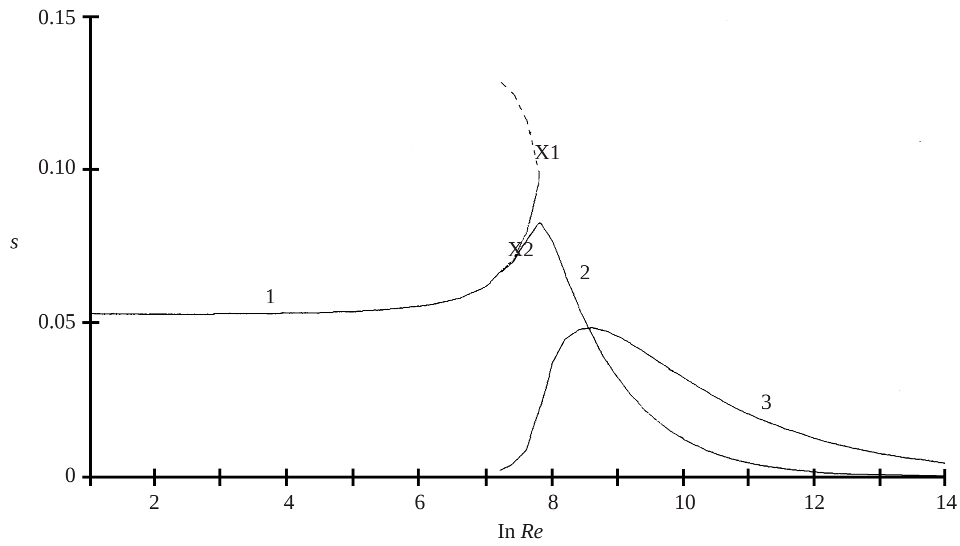

FIGURE 2. Curve 1 for real values of s extends up to the point X1, where $Re = 2283$. The dotted line is the extension of the curve as it moves through the turning point X1. Curve 2 is for the positive real part of the complex values of s for $Re > 1226$ and begins at the point X2. The complex roots form conjugate pairs, and curve 3 is for the positive imaginary part of s.

on the free surface of the drop at sufficiently high Reynolds numbers, as suggested by the analysis in § 2.

The distance between the contact line and its static position tends to zero as a damped oscillation. When s is real, the approach is monotonic and any difference between advancing and receding static angles is irrelevant. With an oscillatory approach, however, both advancing and receding motions occur, so that any contact angle hysteresis will act to suppress the oscillation when its amplitude is sufficiently small. Similar difficulties appear in the analysis of a plate oscillating through a liquid surface where there is contact angle hysteresis (Young & Davis 1987), and of inviscid capillary waves produced by an oscillating plate (Hocking 1987).

For large Reynolds numbers the numerical solution indicates that q is large so that in (3.16) $\tan qh_e \sim 1$ except where $|q|\ (1-x) \sim 1$. Hence, the dominant term of the solution of (3.16) satisfies

$$\frac{\mathrm{d}^3\hat{h}_1}{\mathrm{d}x^3} = -\frac{xq^3h_e}{1-qh_e}. \tag{3.20}$$

The boundary conditions in (3.16) still hold. The leading terms in the solution of (3.20) give terms in $\mathrm{d}h_1/\mathrm{d}x$ of order q and q^2 and the matching condition (3.17) to this order becomes

$$-\frac{2}{s} = \frac{\mathrm{d}\hat{h}_1}{\mathrm{d}x} \quad \text{at} \quad x = 1. \tag{3.21}$$

The right-hand side of (3.20) is regular. The logarithmic term in (3.17) arises from the solution of (3.16) for $|q|\ (1-x) \sim 1$. Successive integrations of (3.20) are straightforward and the application of the boundary condition shows that

$$-\frac{2}{s} = \tfrac{1}{15}q^2 + \tfrac{1}{3}\mathrm{i}q. \tag{3.22}$$

Since $q^2 = s/\varepsilon$, the leading terms in the expansion of s in terms of ε are given by

$$s = \mathrm{i}(30\varepsilon)^{1/2} + \tfrac{5}{2}(30\varepsilon^2)^{1/4}\exp(-\tfrac{1}{4}\mathrm{i}\pi)$$

and the numerical values of these terms are close to the calculated ones shown in figure 2 for $Re > \mathrm{e}^{10}$. It should be noted that the non-dimensionalization makes ε proportional to the kinematic viscosity ν, so the damping rate is proportional to $\sqrt{\nu}$, as expected.

The transition from monotonic to oscillatory behaviour occurs when $\varepsilon = (1226)^{-1}$. For a water drop the material parameters are density $\rho = 1\,\mathrm{g\,cm^{-3}}$, surface tension $\sigma = 70\,\mathrm{dyn\,cm^{-1}}$, and kinematic viscosity $\nu = 10^{-2}\,\mathrm{cm^2\,s^{-1}}$. For a width/height aspect ratio of $2a_0/h_0 = 10$, then h_0 must exceed 1.1 cm for the oscillatory behaviour to be displayed, but the Bond number $\rho g a_0^2/\sigma$ is then over 400, so hydrostatic effects are important. (Of course, this is not an issue in microgravity environments.) For an aspect ratio of 5, however, transition occurs when $h_0 = 0.07$ cm, the width of the drop is then 0.35 cm and the Bond number is only 0.4, so the neglect of hydrostatic effects is justified. In a microgravity environment, the reduction of the Bond number condition allows the theory to extend to larger drops and smaller aspect ratios.

4. Drop on an oscillating plate

The motion of a drop on an oscillating plate is governed by equations similar to those presented in §2. Suppose that the plate moves horizontally with velocity $2U\cos\omega t'$. U provides the velocity scale, which was unprescribed in the spreading problem of §3, and, as well as the time scale $a_0 U$, the frequency of the oscillation provides another time scale $1/\omega$. In addition, the drop is no longer symmetrical, with edges at $x = \pm a(t) + b(t)$. Because there is now an imposed velocity scale, the capillary number cannot be scaled out of the equations. With the same scalings as in (2.6), equations (2.9) become

$$\frac{\partial^3\psi}{\partial z^2} = -\frac{1}{C}\frac{\partial^3 h}{\partial x^3} + \frac{1}{\varepsilon}\left(\frac{\partial^2\psi}{\partial z\partial t} + \frac{\partial\psi}{\partial z}\frac{\partial^2\psi}{\partial x\partial z} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial z^2}\right), \tag{4.1a}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}[\psi(x, h(x,t), t)] = 0, \qquad \int_{-a+b}^{a+b} h\,\mathrm{d}x = \tfrac{2}{3}, \tag{4.1b}$$

and the boundary conditions (2.10) become

$$\psi = 0, \qquad \frac{\partial\psi}{\partial z} = \lambda\frac{\partial^2\psi}{\partial z^2} + 2\cos\Omega t \quad \text{at} \quad z = 0; \qquad \frac{\partial^2\psi}{\partial z^2} = 0 \quad \text{at} \quad z = h(x,t), \tag{4.2a}$$

$$h = 0, \qquad \frac{\partial h}{\partial x} = \mp 1 \quad \text{at} \quad x = \pm a(t) + b(t). \tag{4.2b}$$

The parameters C, ε and, Ω are defined by

$$C = \frac{\rho\nu U}{\sigma}\left(\frac{a_0}{h_0}\right)^3, \quad \varepsilon = \frac{\nu}{U h_0}\left(\frac{a_0}{h_0}\right), \quad \Omega = \frac{\omega a_0}{U}. \tag{4.3}$$

If Ω is of unit order, the unsteady and quadratic inertial terms are of the same order, so the inclusion of inertia would require the solution of a nonlinear equation. In the present problem, if $\Omega \gg 1$, the time-derivative term forced by the oscillation of the plate dominates the quadratic terms, so, to leading order, a linear problem is relevant. In what follows $2\cos\Omega t$ is replaced by $\exp(\Omega t)$ and the real part is understood.

In the central region, away from the edges of the drop, the leading terms in powers of C of the height of the drop and the stream function have the forms

$$h = h_s + Ch_1 + C\exp(\mathrm{i}\Omega t)h_2, \quad \psi = \psi_1 + \exp(\mathrm{i}\Omega t)\psi_2, \tag{4.4}$$

where the quasi-static height of the drop is given by

$$h_s = \frac{a^2 - (x-b)^2}{2a^3}. \tag{4.5}$$

The non-oscillatory part of the solution is the same as when there is no inertia and no oscillation of the plate, except that the centre of the drop is displaced to $x = b$. From (4.1) and (4.2), it follows that

$$\frac{\partial^3 h_1}{\partial x^3} = \frac{3}{a}\frac{\mathrm{d}a}{\mathrm{d}t}\frac{x-b}{h_s^2}, \qquad \frac{\partial h_1}{\partial x} = 0 \quad \text{at} \quad x-b=0, \qquad h_1 = 0 \quad \text{at} \quad x-b = \pm a. \tag{4.6}$$

The solution follows that given in Hocking (1982), and shows that, near the edges of the drop,

$$\frac{\partial h_1}{\partial x} \sim \pm 3a^4\frac{\mathrm{d}a}{\mathrm{d}t}[-\ln(a \pm (b-x)) + \ln 2a - 3] \quad \text{as} \quad x-b \to \pm a. \tag{4.7}$$

For the oscillatory part of the solution, (4.1) and (4.2) give

$$\frac{\partial^3 \psi_2}{\partial z^2} - \frac{\mathrm{i}\Omega}{\varepsilon}\frac{\partial \psi_2}{\partial z} = -\frac{\partial^3 h_2}{\partial x^2}, \tag{4.8a}$$

$$\psi_2 = 0, \qquad \frac{\partial \psi_2}{\partial z} = 1 \quad \text{at} \quad z = 0, \qquad \frac{\partial^2 \psi_2}{\partial z^2} = 0 \quad \text{at} \quad z = h_s, \tag{4.8b}$$

$$\frac{\partial \psi_2}{\partial x} = -\frac{\mathrm{i}\beta\Omega(x-b)}{a^3} \quad \text{at} \quad z = h_s, \tag{4.8c}$$

where $b = \beta\exp(\mathrm{i}\Omega t)$. The last of these conditions can be written in the equivalent form

$$\psi_2 = \mathrm{i}\beta\Omega h_s \quad \text{at} \quad z = h_s. \tag{4.9}$$

The solution of (4.8a) is elementary, and, when the boundary conditions are applied,

$$\psi_2(x, h_s(x)) = \frac{h_s}{q^2}\left(1 - \frac{\tanh qh_s}{qh_s}\right)\frac{\partial^3 h_2}{\partial x^3} + \frac{\tanh qh_s}{q}, \tag{4.10a}$$

where

$$q^2 = \mathrm{i}\Omega/\varepsilon; \tag{4.10b}$$

q plays the same role as q in (2.14). From (4.9), the equation satisfied by this component of the height of the drop is found to be

$$\frac{\partial^3 h_2}{\partial x^3} = q^2\left[1 + (\mathrm{i}\beta\Omega - 1)\frac{qh_s}{qh_s - \tanh qh_s}\right]. \tag{4.11}$$

In the non-oscillatory part of the solution, ψ_1 is odd and h_2 even about the midpoint of the drop, whereas the oscillatory part, ψ_2, is even and h_2 is odd. The conditions to be applied in solving (4.11) are that

$$h_2 = 0, \qquad \frac{\partial^2 h_2}{\partial x^2} = 0 \quad \text{at} \quad x-b=0, \qquad h_2 = 0 \quad \text{at} \quad x-b=a. \tag{4.12}$$

Because of the singularity at the edge, where $h_s = 0$, it is convenient to write (4.11) in the form

$$\frac{\partial^3 h_2}{\partial x^3} = q^2 \left[1 + (\mathrm{i}\beta\Omega - 1)\left(\frac{3}{q^2 h_s^2} + \frac{6}{5} + F(x) \right) \right], \tag{4.13}$$

where

$$F(x) = \frac{q h_s}{q h_s - \tanh q h_s} - \frac{3}{q^2 h_s^2} - \frac{6}{5}. \tag{4.14}$$

All the terms in (4.13), except for the last one, can then be integrated exactly. $F(x)$ is zero at the edge, where $h_s = 0$, so there is no difficulty in performing a numerical integration. If a function $G(x)$ is defined by

$$\frac{\mathrm{d}^3 G}{\mathrm{d}x^3} = F(x), \qquad G = \frac{\mathrm{d}G}{\mathrm{d}x} = \frac{\mathrm{d}^2 G}{\mathrm{d}x^2} = 0 \quad \text{at} \quad x - b = 0, \tag{4.15}$$

the values of $G(a)$ and of $G'(a) = \mathrm{d}G/\mathrm{d}x$ at $x - b = a$ can be found by numerical integration. To find the slope of the drop profile as the edge is approached from the central region, the various components of the solution combine to show that, as $x \to a + b$,

$$-\frac{\partial h}{\partial x} \sim \frac{1}{a^2} - 3Ca^4[-\ln(a+b-x) + \ln 2a - 3]\left(\frac{\mathrm{d}a}{\mathrm{d}t} + \frac{\mathrm{d}b}{\mathrm{d}t} - \exp(\mathrm{i}\Omega t) \right)$$
$$-C\exp(\mathrm{i}\Omega t) q^2[\tfrac{1}{3}a^2 + (\mathrm{i}\beta\Omega - 1)(\tfrac{2}{5}a^2 + G'(a) - G(a)/a)]. \tag{4.16}$$

In the slip region near the edge of the drop at $x = a + b$, new variables are defined by

$$h = \lambda H, \quad z = \lambda Z, \quad \psi = \lambda \Psi, \quad a + b - x = \lambda X, \tag{4.17}$$

and a similar expansion to that used in the central region is given by

$$H = X + CH_1(X) + C\exp(\mathrm{i}\Omega t) H_2(X),$$
$$\Psi = \Psi_1 + \exp(\mathrm{i}\Omega t)\Psi_2. \tag{4.18}$$

The scaled versions of (4.1) and (4.2) are

$$\left.\begin{aligned} &\frac{1}{\lambda^2}\left(\frac{\partial^3 \Psi_1}{\partial Z^3} - \frac{\partial^3 H_1}{\partial X^3} \right) = 0, \quad \Psi_1 = 0, \quad \frac{\partial \Psi_1}{\partial Z} = \frac{\partial^2 \Psi}{\partial Z^2} \quad \text{at} \quad Z = 0, \\ &\frac{1}{\lambda^2}\left(\frac{\partial^3 \Psi_2}{\partial Z^3} - \frac{\partial^3 H_2}{\partial X^3} \right) = \frac{\mathrm{i}\Omega}{\varepsilon}\frac{\partial \Psi_2}{\partial Z}, \quad \Psi_2 = 0, \quad \frac{\partial \Psi_2}{\partial Z} = \frac{\partial^2 \Psi_2}{\partial Z^2} + 1 \quad \text{at} \quad Z = 0. \end{aligned}\right\} \tag{4.19}$$

The conditions to be applied at $Z = H$ are

$$\frac{\partial^2 \Psi_1}{\partial Z^2} = \frac{\partial^2 \Psi_2}{\partial Z^2} = 0, \qquad \frac{\partial \Psi_1(X,H)}{\partial X} = \frac{\mathrm{d}a}{\mathrm{d}t}, \qquad \frac{\partial \Psi_2(X,H)}{\partial X} = \mathrm{i}\beta\Omega, \tag{4.20}$$

and, to leading order, $H = X$. Provided the parameters are such that

$$\frac{\lambda^2 \Omega}{\varepsilon} \ll 1, \tag{4.21}$$

the inertial effects are not significant within the slip region. For all realistic values of the slip coefficient, this condition is satisfied. The solution in the edge region then follows exactly as for the inertialess spreading-drop problem of Hocking (1982), and the slope of the drop profile away from the edge is given by

$$\frac{\partial H}{\partial X} \sim 1 + 3C(\ln X - \ln 3 + 1)\left(\frac{\mathrm{d}a}{\mathrm{d}t} + \frac{\mathrm{d}b}{\mathrm{d}t} - \exp(\mathrm{i}\Omega t)\right) \quad \text{as} \quad X \to \infty. \tag{4.22}$$

As in the spreading of a non-oscillating drop without inertia, an intermediate region of thickness of order $1/\ln(\lambda)$ is needed. An equivalent analysis to that given in Hocking (1982) can be performed, and, as in that paper, the matching across the intermediate layer shows that

$$\left(-\frac{\partial h}{\partial x}\right)^3_{x\to a+b} \sim \left(\frac{\partial H}{\partial X}\right)^3_{X\to\infty}. \tag{4.23}$$

The variable logarithmic terms on either side cancel, and the other terms give

$$\frac{1}{a^6} - 1 + 9C\frac{\mathrm{d}a}{\mathrm{d}t}\left[\ln\left(\frac{3\lambda}{2a}\right) + 2\right] - C\exp(\mathrm{i}\Omega t)\frac{q^2}{a^2} + C\exp(\mathrm{i}\Omega t)(\mathrm{i}\beta\Omega - 1)$$
$$\times\left[9\ln\left(\frac{3\lambda}{2a}\right) - \frac{q^2}{a^4}\left(\tfrac{6}{5}a^2 + 3G'(a) - 3G(a)/a\right)\right] = 0. \tag{4.24}$$

A similar equation holds for the matching near the edge at $x = -a + b$, except that the signs of all the terms with the exponential factor are reversed. This is because the oscillatory forcing produces a change in the drop profile that is antisymmetric about the midpoint of the drop, while the spreading is otherwise symmetric. From the symmetric terms, it follows that the spreading velocity of the drop is given by

$$9C\frac{\mathrm{d}a}{\mathrm{d}t}\left[\ln\left(\frac{2a}{3\lambda}\right) - 2\right] = \frac{1}{a^6} - 1. \tag{4.25}$$

This result shows that the width of the drop increases or decreases exactly as when the plate is stationary, but the midpoint of the drop is no longer stationary. The antisymmetric terms show that the midpoint of the drop has a velocity given by

$$\frac{\mathrm{d}b}{\mathrm{d}t} = \exp(\mathrm{i}\Omega t) + \frac{q^2\exp(\mathrm{i}\Omega t)}{9a^2\ln(3\lambda/2a) - 3(q^2/a^2)\left[\frac{2}{5}a^2 + G'(a) - G(a)/a\right]}. \tag{4.26}$$

In the absence of inertia, $q = 0$ and the drop motion is a combination of a rigid oscillation with the plate and the mean spreading that occurs as though there were no oscillation. The spreading is unaffected by inertia, in the parameter range considered here, but the oscillatory motion is influenced by inertia, in a manner determined by (4.26).

The midpoint speed for any given value of a is a function of two parameters, defined by

$$\Lambda = \lambda/a, \qquad P = \Omega/sa^2, \quad \text{with} \quad q^2 = \mathrm{i}Pa^2. \tag{4.27}$$

Scaled values of G and G' are defined by $G(a) = a^3g(1)$, $G'(a) = a^2g'(1)$, with g independent of a and defined, from (4.13) and (4.14), by the solution of

$$\frac{\mathrm{d}^3g}{\mathrm{d}x^3} = \frac{(\mathrm{i}P)^{1/2}\hat{h}}{(\mathrm{i}P)^{1/2}\hat{h} - \tanh[(\mathrm{i}P)^{1/2}\hat{h}]} - \frac{3}{\mathrm{i}P\hat{h}^2} - \frac{6}{5}, \quad \hat{h} = \tfrac{1}{2}(1 - x^2), \tag{4.28a}$$

$$g = \frac{\mathrm{d}g}{\mathrm{d}x} = \frac{\mathrm{d}^2g}{\mathrm{d}x^2} = 0 \quad \text{at} \quad x = 0. \tag{4.28b}$$

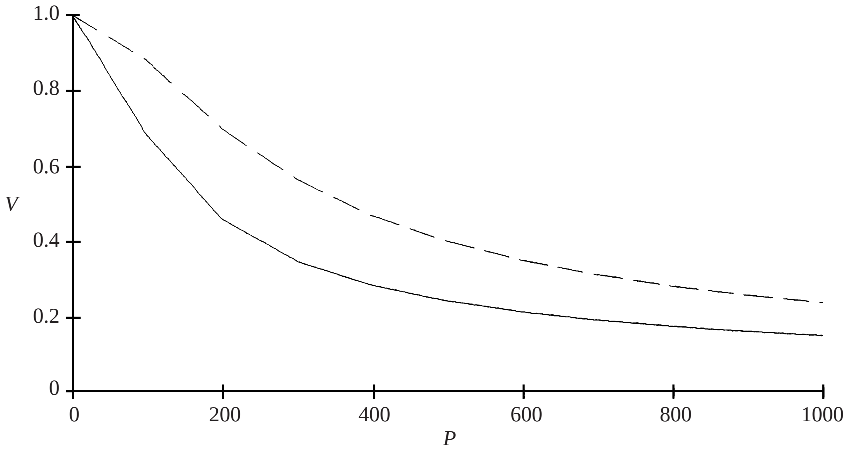

FIGURE 3. Speed of the midpoint of the drop: ——, $\Lambda = 10^{-5}$; – – –, $\Lambda = 10^{-10}$.

Then, the magnitude V and phase ϕ of the midpoint velocity are given by

$$\frac{\mathrm{d}b}{\mathrm{d}t} = V \exp(\mathrm{i}\Omega t - \mathrm{i}\phi), \tag{4.29a}$$

$$V \exp(-\mathrm{i}\phi) = 1 + \frac{\mathrm{i}P}{9\ln(\frac{3}{2}\Lambda) - 3\mathrm{i}P\left[\frac{2}{5} + g'(1) - g(1)\right]}. \tag{4.29b}$$

Calculated values of V as a function of P for two different values of Λ are shown in figure 3. The velocity of the edge of the drop is given by adding this velocity to that of the steady spreading of the edge of the drop. Relative to the monotonic spreading of the drop, the contact line moves with the plate when $P = 0$ or $\Lambda = 0$. As the frequency of the oscillation increases, or as the slip coefficient is increased, the speed of the contact line is reduced. Compare this to the solution found in § 2.3 for a thin film on an oscillating plate.

The asymptotic velocity for large Reynolds numbers can be found analytically. For large P, equation (4.28) has the form

$$\frac{\mathrm{d}^3 g}{\mathrm{d}x^3} = -\frac{1}{5} + \frac{2}{(\mathrm{i}P)^{1/2}(1 - x^2)}, \tag{4.30}$$

and the integration can be performed exactly. The asymptotic form of (4.29) for large P is given by

$$V \exp(-\mathrm{i}\phi) \sim \frac{3}{(\mathrm{i}P)^{1/2}} - \frac{9(\ln 3\Lambda/2 + 1)}{\mathrm{i}P}. \tag{4.31}$$

The values of V for $P = 1000$ given by this formula are within 15% of the exact values calculated from (4.28) and shown in figure 3.

The phase ϕ is shown in figure 4 for the same two values of Λ. It reaches a value approximately equal to 0.3π for $P = 500$ and remains close to that value for P as large as 10^4. The asymptotic value of ϕ from (4.31) is given by

$$\tan\phi = 1 + \frac{K}{P^{1/2}}, \tag{4.32}$$

where $K = 42.9$ for $\Lambda = 10^{-5}$, $K = 91.7$ for $\Lambda = 10^{-10}$. Hence the limiting value of $\pi/4$ is not approached until values of P as large as 10^5 or 10^6.

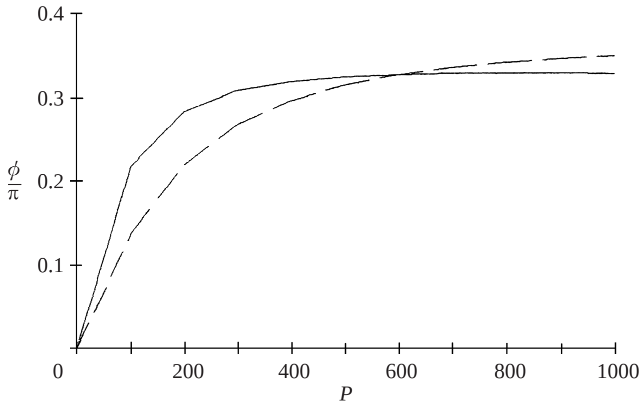

FIGURE 4. Phase of the midpoint speed: ——, $\Lambda = 10^{-5}$; – – –, $\Lambda = 10^{-10}$.

For the range of parameters considered here, inertia does not affect the mean spreading of the drop. The amplitude of the contact-line motion superimposed on this spreading is equal to the amplitude of the plate oscillation when there is no inertia. Inertia reduces the amplitude of the contact-line motion, and causes this motion to lag behind that of the plate.

5. Contact angles

In problems involving moving contact lines, the dynamic behaviour of the contact angle is usually expressed by its dependence on the speed of the contact line. For the problem considered in § 3, the slope of the leading term in the outer solution at the edge is given by

$$-\frac{\partial}{\partial x}\left(\frac{a^2-x^2}{2a^3}\right)_{x=a} = \frac{1}{a^2} = 1 - 2\alpha\exp(-st) = 1 + \frac{2}{s}\frac{\mathrm{d}a}{\mathrm{d}t}, \tag{5.1}$$

from (3.5). If this slope is taken to be the scaled apparent contact angle, it follows that for the approach to the terminal state, this contact angle can be expressed either in terms of the contact-line speed or of the drop width. From (5.1) and (3.19), these relations are, for the angle/speed relation,

$$\frac{\theta_{app}}{\theta_s} = 1 + \frac{2}{s}\frac{\mathrm{d}a}{\mathrm{d}t}, \quad \frac{2}{s} = 3\ln(1/\lambda) - F(q), \tag{5.2}$$

and for the angle/width relation

$$\frac{\theta_{app}}{\theta_s} = 1 - 2(a-1). \tag{5.3}$$

The calculated values of s are shown in figure 2 and the effect of inertia in (5.2) enters through the parameter q. For $Re < 1226$, s increases only slightly as inertia increases. For $Re > 1226$, s is a complex quantity, and it is more convenient to use (5.2) to evaluate the contact angle. In real terms, a and $\mathrm{d}a/\mathrm{d}t$ are not proportional, and there is no simple contact angle/contact speed relation. However, the contact angle can be expressed as a linear function of the contact-line speed and acceleration,

$$\frac{\theta_{app}}{\theta_s} = 1 + \frac{2}{|s|^2}\left(2s_r\frac{\mathrm{d}a}{\mathrm{d}t} + \frac{\mathrm{d}^2a}{\mathrm{d}t^2}\right), \tag{5.4}$$

where s_r is the real part of s.

For the drop spreading on an oscillating plate discussed in § 4, the mean spreading velocity is unaffected by the oscillation. Without inertia, the whole drop oscillates rigidly with the plate. The oscillatory component of the edge velocity (4.29) is the response of the drop to the forced motion of the plate and inertia reduces its magnitude as well as producing a phase lag. For the spreading component, the apparent contact angle can be deduced from the leading hydrostatic term in the drop profile, so that, from (4.16) and (4.25), the spreading component θ_{spr} is given by

$$\left(\frac{\theta_{spr}}{\theta_s}\right)^3 - 1 = 9C\frac{\mathrm{d}a}{\mathrm{d}t}\left[\ln\left(\frac{2a}{3\lambda}\right) - 2\right]. \tag{5.5}$$

It is independent of inertia in the parameter range considered, and so is the same as when the plate is stationary. For the oscillatory part of the solution, there is no leading hydrostatic term, and the slope has a logarithmic singularity as the edge is approached, which is matched with the inner region. If we define the oscillatory component of the contact line, θ_{osc}, as the slope measured at some distance ℓ from the edge, where $1 \gg \ell \gg \lambda$, then, from (4.16) and (4.26),

$$\frac{\theta_{osc}}{\theta_s} = 3C\ln\left(\frac{\ell}{\lambda}\right)\left[\frac{\mathrm{d}b}{\mathrm{d}t} - \exp(\mathrm{i}\Omega t)\right]. \tag{5.6}$$

The numerical factors have been absorbed into the definition of ℓ or λ. In real terms, with the plate speed equal to $2\cos\Omega t$, the oscillatory contact angle is given by

$$\frac{\theta_{osc}}{\theta_s} = 6C\ln\left(\frac{\ell}{\lambda}\right)[V\cos(\Omega t - \phi) - \cos\Omega t], \tag{5.7}$$

where V and ϕ are defined by (4.29). For zero inertia, $V = 1$ and $\phi = 0$, so that $\theta_{osc} = 0$. As inertia is increased, V decreases and the magnitude of θ_{osc} increases, but it is out of phase with both the contact-line speed and the plate speed. For large inertia, $V \to 0$, and θ_{osc} is proportional to the plate speed. An alternative form (5.7) can be found, in terms of the contact-line speed and acceleration,

Inertial effects on dynamic contact angles are discussed by Cox (1998). The contact angle is not assumed to be small, and he extends his earlier analysis (Cox 1986) to include inertial effects by examining the local wedge solution in the intermediate region. He determines the contact angle/velocity relationship in two cases, one when inertial effects modify the slip region and the other when they only affect the wedge region. No attempt is made in his paper to determine a complete solution by matching, as only the vicinity of the edge is considered. In both cases, he predicts that the apparent and static contact angles are related by an equation of the form

$$g(\theta) - g(\theta_s) = C\ln(k),$$

where

$$\begin{aligned} k &= 1/\lambda, \quad \text{when} \quad Re \gg 1/\lambda \gg 1, \\ k &= Re, \quad \text{when} \quad 1/\lambda \gg Re \gg 1, \end{aligned} \tag{5.8}$$

where g is a known function, θ and θ_s are the apparent and static contact angles, and λ is a non-dimensional slip length. For a typical speed U and length a, the capillary number $C = \rho\nu U/\sigma$, and the Reynolds number $Re = Ua/\nu$. These results have been tested experimentally by Stoev, Ramé & Garoff (1999) for a vertical plate moving into a pool of liquid for moderate Reynolds numbers. The prediction of Cox that inertia will decrease the apparent contact angle is verified.

For small angles, Cox's predictions become

$$\theta^3 - \theta_s^3 = 3.88C \ln(k), \tag{5.9}$$

which is nearly the same as the corresponding result when $Re \ll 1$,

$$\theta^3 - \theta_s^3 = 3C \ln(1/\lambda). \tag{5.10}$$

The contact angle/speed relationship (5.2) for a thin drop as it approaches equilibrium, found in § 3, differs from (5.8). In Cox's results, $\ln(k)$ contains a term $\ln(1/\lambda)$ when $R \gg 1/\lambda$, and a term $\ln(Re)$ when $R \ll 1/\lambda$. The present results contain the $\ln(1/\lambda)$ term with $R \ll 1/\lambda$ and appear to suggest a possible behaviour dominated by $\ln R$ when $R \gg 1/\lambda$.

It is not surprising that the present results differ from those of Cox. His analysis assumes (*a*) an inviscid region above a viscous boundary layer, even close to the contact line, (*b*) the existence of an intermediate region, where a wedge-like solution holds, (*c*) monotonic motion. In the analysis presented here (*a*) in the terminal stage, vorticity at the base of the drop has spread throughout the drop, (*b*) no intermediate region is present, (*c*) for the larger values of *Re* considered, the motion is oscillatory. For these reasons, the effect of inertia on the terminal stage in the spreading of a drop cannot meaningfully be related to Cox's results. For the drop on an oscillating plate considered in § 4, the spreading part of the motion is unaffected by inertia. The oscillatory part cannot be related to Cox's results, since he did not examine non-monotonic motions.

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